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A New Theory of Elementary Matter Part IV: Two-Particle Systems: The Particle–Antiparticle Pair and Hydrogen

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Abstract

The general theory developed thus far (Sachs, 1971b, c, d) is applied to two-particle systems. An exact bound state solution of the nonlinear field equations of this theory for a particle–antiparticle pair is demonstrated. From the Lagrangian formalism, this solution is shown to predict all of the experimental facts that are conventionally interpreted in terms of 'pair annihilation': (1) the energy-momentum four-vector (and each of the four components, separately) are zero, compared with the energy, $2mc^2$, of the state when the particle and antiparticle are (asymptotically) free and (2) the dynamical properties of this state of positronium make it appear in experimentation as two distinguishable currents, correlated with a 90° phase difference and polarised in a plane that is perpendicular to the direction of propagation of interaction with other charged matter. The latter features are conventionally interpreted as the two photons which are produced in the annihilation event—however, there are no photons in this theory. The spectral distribution of blackbody radiation is then derived from the properties of an ideal gas of such pairs, in their ground states of null energy-momentum, as observed in a finite cavity.

The properties of the *closed* electron–proton system are considered and the entire hydrogen spectrum is derived—including the Lamb splitting. The correct lifetimes of the excited hydrogenic states are then derived by considering the radiating hydrogen gas to be immersed in the ideal gas of pairs, that explained blackbody radiation.

1. *The Particle–Antiparticle Pair*

Throughout the preceding theoretical development, we have studied the features of a formalism, for either the entire closed system of many interacting components, or the 'one body approximation', where we considered the coupling of a single matter component to an averaged background field (or no background field), to represent the remainder of the closed system. In this section, and in the following one, we will investigate two very important 'two-body systems'—assuming again that the rest of the closed system can either be neglected or that it can be represented by an averaged background field.

With the assumed uncoupling of the two-body system from the rest of the closed system (which is nevertheless *there*!) we will consider, in this section, a solution of the coupled field equations [Part III, equation (3.8)] for the particle–antiparticle pair, that has particularly important physical consequences. This *exact* solution will be shown to relate to all of the experimental facts that are conventionally interpreted as 'pair annihilation'. In the following section, we will study the electron–proton system—showing that the entire hydrogen spectrum (including the Lamb splitting) emerges as a prediction of the formalism. Combining the *exact* prediction of this section with the results on the hydrogen spectrum, it will then be demonstrated in the following section that the correct 'lifetimes' of the excited states of hydrogen must also follow.

1.1. *The Field Equations for an Electron–Positron Pair* (Sachs & Schwebel, 1961)

Let us now consider the field equations [Part III, equation (3.8)] to describe the bound electron–positron pair, *neglecting the rest of the universe.* Since each of these components have the same mass parameter, the field equations have the following form for the particle–antiparticle pair:

$$\{\gamma_\mu \partial^\mu - \mathscr{I}(e^+) + \lambda\}\psi^{(e^-)} = 0 \tag{1.1.1a}$$

$$\{\gamma_\mu \partial^\mu - \mathscr{I}(e^-) + \lambda\}\psi^{(e^+)} = 0 \tag{1.1.1b}$$

where $\psi^{(e^+)} = C\psi^{(e^-)}$, and

$$C = \gamma_2 K_0 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} K_0$$

is the 'charge conjugate operator'. K_0 is the operator corresponding to taking the complex conjugate of the function to which it applies.

As we have seen in Part III, the interaction coupling term, $\mathscr{I}$, in electrodynamics, has two terms:

$$\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2$$

where

$$\mathscr{I}_1(e^\pm)\psi^{(e^\mp)} = [e^\pm e^\mp \gamma_\nu \int \bar{\psi}^{(e^\pm)} \gamma_\nu \psi^{(e^\pm)} S(x - x')\, d^4x']\psi^{(e^\mp)} \tag{1.1.2a}$$

$$\mathscr{I}_2(e^\pm)\psi^{(e^\mp)} = -ig_M e^\mp \sum_{\alpha=1}^{2} a_\alpha(\varphi_\alpha^{(e^\pm)\dagger} \cdot \Gamma_\alpha - \gamma_0 \Gamma_\alpha{}^\dagger \gamma_0 \cdot \varphi_\alpha^{(e^\pm)})\psi^{(e^\mp)} \tag{1.1.2b}$$

It is clear that the coupled equations (1.1.1) satisfy the requirement of covariance with respect to special relativity, and covariance with respect to the interchange of the matter field variables $\psi^{(e^-)}$ and $\psi^{(e^+)}$.

Consider now the special case where the particle and antiparticle fields correspond to the same state of motion. In this case, the source fields in the electromagnetic field equations for the particle and antiparticle must also

be the same. This is in the sense of being characterized by the same constants of the motion. Denoting the latter by the index n, the spinor form of the electromagnetic equations (Parts II and III) for the electron and positron fields are respectively:

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(e^-)} = -e\bar{\psi}_n^{(e^-)} \Gamma_\alpha \psi_n^{(e^-)} \tag{1.1.3a}$$

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(e^+)} = e\psi_n^{(e^+)} \Gamma_\alpha \psi_n^{(e^+)} \tag{1.1.3b}$$

In these relations we have used the notation

$$e^+ = -e^- = e \tag{1.1.3c}$$

Substituting $C\psi_n^{(e^-)}$ for $\psi_n^{(e^+)}$ in equation (1.1.3b), we have

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(e^+)} = e\bar{\psi}_n^{(e^-)} \Gamma_\alpha \psi_n^{(e^-)} \tag{1.1.3d}$$

This follows from the fact that substitution of $C\psi^{(e^-)}$ for $\psi^{(e^+)}$ in the current density terms $\bar{\psi}^{(e^+)} \gamma_\mu \psi^{(e^+)}$ leads to their being equal to $\psi^{(e^-)} \gamma_\mu \psi^{(e^-)}$, and the fact that Γ_α are four-dimensional matrices that are linear combinations of the Dirac matrices γ_μ.

It then follows that *when the electron and positron fields correspond to the same state of motion*, the sum of equations (1.1.3a) and (1.1.3b) yields the following electromagnetic equations *for the pair*:

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(\text{pair})} = 0 \tag{1.1.4}$$

where

$$\varphi_\alpha^{(\text{pair})} = \varphi_\alpha^{(e^-)} + \varphi_\alpha^{(e^+)} \tag{1.1.5}$$

The solution $\varphi_\alpha^{(\text{pair})}$ for the pair *in this particular state of motion* represents the electromagnetic field of influence that is exerted by the pair, *as a unit*, on other charged matter. But according to the interpretation of these equations as an identity (that follows from the presented theory (Parts I and II)) the source field on the right-hand side of equation (1.1.4) being zero means that $\varphi_\alpha^{(\text{pair})}$ is also identically zero. It is then concluded that when the interacting components of the particle–antiparticle pair are in the same state of motion, this bound two-particle system will not couple, *as a unit*, to other charged matter. Nevertheless, each of the constituent interacting components of this pair does couple, separately, to other charged matter, since each has a nonzero field (of its own). It also follows that the electron and positron field components separately maintain their inertial properties Consequently, the field equations (1.1.4) for the pair does incorporate a substructure which would not allow the solutions to be interpreted in terms of source-free electromagnetic radiation.

Finally, it should be noted that when the particle and antiparticle are in the same state of motion, equation (1.1.3d) in equation (1.1.2) implies that (under these special circumstances)

$$\mathscr{I}(e^-)\psi^{(e^+)} = \mathscr{I}(e^+)\psi^{(e^-)} \tag{1.1.6}$$

1.2. *An Exact Solution of the Field Equations*

It will now be demonstrated that when $a_1 = -a_2$ in the coupling term $\mathscr{I}_2$, then with the electron and positron fields in the same state of motion, an exact solution of equation (1.1.1a) is

$$\psi^{(e^+)} = -\psi^{(e^-)} = \begin{pmatrix} e^{-i\lambda t} \\ 0 \\ 0 \\ e^{i\lambda t} \end{pmatrix} \equiv \psi \tag{1.2.1}$$

Substituting the function (1.2.1) into the electromagnetic field equations (1.1.3), we have

$$\sigma_\mu \partial^\mu \varphi_1^{(e^+)} = e\bar{\psi}\Gamma_1 \psi = -8\pi ie \begin{pmatrix} 1 \\ -\exp(2i\lambda t) \end{pmatrix} \tag{1.2.2a}$$

$$\sigma_\mu \partial^\mu \varphi_2^{(e^+)} = e\bar{\psi}\Gamma_2 \psi = 8\pi ie \begin{pmatrix} -\exp(-2i\lambda t) \\ 1 \end{pmatrix} \tag{1.2.2b}$$

Using the integral representation (Sachs, 1971a) for the solutions of equations (1.2.2), we obtain the following result:

$$\varphi_1(x)^{(e^+)} = -\frac{8\pi ie}{(2\pi)^4} \int \exp[ik^\mu(x_\mu - x_\mu')] \frac{(-i\bar{\sigma}_\beta k^\beta)}{k_\alpha k^\alpha} \begin{pmatrix} 1 \\ -\exp(2i\lambda t') \end{pmatrix} d^4k d^4x' \tag{1.2.3a}$$

$$\varphi_2(x)^{(e^+)} = \frac{8\pi ie}{(2\pi)^4} \int \exp[ik^\mu(x_\mu - x_\mu')] \frac{(-i\bar{\sigma}_\beta k^\beta)}{k_\alpha k^\alpha} \begin{pmatrix} -\exp(-2i\lambda t') \\ 1 \end{pmatrix} d^4k d^4x' \tag{1.2.3b}$$

Carrying out these integrations, we find that

$$\varphi_1^{(e^+)}(x) = \frac{4\pi e}{\lambda} \begin{pmatrix} 0 \\ \exp(2i\lambda t) \end{pmatrix} \tag{1.2.4a}$$

$$\varphi_2^{(e^+)}(x) = \frac{4\pi e}{\lambda} \begin{pmatrix} \exp(-2i\lambda t) \\ 0 \end{pmatrix} \tag{1.2.4b}$$

The solutions (1.2.3) satisfy the differential equations (1.2.1) while the integrated functions in equation (1.2.4) do not. The apparent difficulty is associated with the treatment of the constant part of the source fields in the integral form of the solutions (1.2.3), and with the associated breakdown of the conditions for the existence of a Fourier transform. Equation (1.2.4), in fact, reveals only a part of the actual solution—the part that is coordinate-dependent. To complete the solution, we add to the spinor (1.2.4a) the constant spinor

$$s_1 = \frac{4\pi e}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.5a}$$

and to the spinor (1.2.4b), the constant spinor

$$s_2 = \frac{4\pi e}{\lambda} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.5b}$$

The resulting spinors, which are now unique particular solutions of the field equations (1.2.2), have the form

$$\varphi_1(x) = \frac{4\pi e}{\lambda}\begin{pmatrix} 1 \\ \exp(2i\lambda t) \end{pmatrix} \tag{1.2.6a}$$

$$\varphi_2(x) = \frac{4\pi e}{\lambda}\begin{pmatrix} \exp(-2i\lambda t) \\ 1 \end{pmatrix} \tag{1.2.6b}$$

These are the electromagnetic spinor variables that are to be inserted into the coupling terms [equation (1.1.2b)], which, in turn, are inserted into the field equations (1.1.1).

A similar situation occurs in the determination of the coupling term $\mathscr{I}_1$ [equation (1.1.2a)]. The integral in this term actually stands for the particular solution A_μ of the equation

$$\Box A_\mu = 4\pi e\bar{\psi}\gamma_\mu\psi \tag{1.2.7}$$

However, as we have seen earlier [Part II (Sachs, 1971c)], the D'Alembertian operator $\Box$ is a product of a (first-order differential) quaternion operator and its conjugate operator, i.e. equation (1.2.7) refers to a doubly iterated equation:

$$\sigma_\mu\partial^\mu(\bar{\sigma}_\mu\partial^\mu A_\nu) = 4\pi\sigma_0(e\bar{\psi}\gamma_\nu\psi) \tag{1.2.7'}$$

Equation (1.2.7) is then equivalent to the pair of first-order differential equations

$$\begin{aligned} \sigma_\mu\partial^\mu\Sigma_\nu &= \sigma_0(e\bar{\psi}\gamma_\nu\psi) \\ \Sigma_\nu &= \bar{\sigma}_\mu\partial^\mu A_\nu \end{aligned} \tag{1.2.7''}$$

It follows that the same technique which was used to solve the spinor equations (1.2.2) can be used to solve equation (1.2.7). The solutions Σ_ν (for each of the four values of ν) are given by the form in equation (1.2.3), with the substitutions

$$\begin{aligned} \varphi_\alpha &\to \bar{\sigma}_\mu\partial^\mu A_\nu \\ e\bar{\psi}\Gamma_\alpha\psi &\to (e\bar{\psi}\gamma_\nu\psi)\,\sigma_0 \end{aligned} \tag{1.2.8}$$

Also, as in the preceding analysis of the electromagnetic equations, the constant spinors, proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, must be added to the integral form of the solutions whenever the source field might be constant. As in the previous case, this addition completes the solution of the (first iteration) of the differential equation (1.2.7') and gives a null solution for a null source—as required.

Once we have solved for $\Sigma_\nu = \bar{\sigma}_\mu\partial^\mu A_\nu$ in this way, the operation must be repeated once more in order to solve for A_ν from the second of the spinor equations (1.2.7''). The form of the solution of this equation is identical with the first form except for the conjugation of all quaternions that appear in equation (1.2.3).

Carrying out this procedure for the specific solution (1.2.1), the following result is obtained for $\mathscr{I}_1$ [equation (1.1.2a)]:

$$\mathscr{I}_1 = \left(\frac{4\pi e^2 i}{2\lambda^2}\right)(\gamma_0 - \gamma_1 \cos 2\lambda t - \gamma_2 \sin 2\lambda t) \tag{1.2.9}$$

Since the solution (1.2.1) in the function $\bar{\psi}\gamma_0\psi$ gives rise to a constant—which in turn yields a null value for the integral

$$\int \bar{\psi}\gamma_0 \psi S(x - x')\, d^4 x'$$

the first term on the right-hand side of equation (1.2.9) had to be derived from the insertion of the constant contributions to the solutions of equation (1.2.7′), as specified above. The second and third parts on the right-hand side of equation (1.2.9) follow from the integration of the right-hand side of equation (1.1.2b), with the insertion of equation (3.6), Part III, for the Green's function $S(x - x')$.

It is readily verified that with the solution (1.2.1) for the electron and positron field, the following relationship is true:

$$(\gamma_1 \cos 2\lambda t + \gamma_2 \sin 2\lambda t)\psi = \gamma_0 \psi \tag{1.2.10}$$

It then follows from the combination of equations (1.2.9) and (1.2.10) that

$$\mathscr{I}_1(e)\psi^{(e)} = 0 \tag{1.2.11}$$

To evaluate the second part of the electrodynamic interaction [equation (1.1.2b)] we substitute the spinor field solutions (1.2.6), for the electromagnetic field intensity, and (1.2.1) for the matter field of the electron or positron. Using also the relationship (1.2.10), we then find that

$$\mathscr{I}_2(e)\psi^{(e)} = -32\pi^2 \left(\frac{e}{\lambda}\right) g_M (a_1 + a_2)\gamma_3 \psi^{(e)} \tag{1.2.12}$$

It is at this stage where we insert $a_1 = -a_2$—thereby fixing the form of the electrodynamic interaction $\mathscr{I}_2$ for all future applications. The following result is then obtained:

$$\mathscr{I}_2 \psi = 0$$

Combining this result with equations (1.2.9) and (1.2.11), it follows that

$$(\mathscr{I}_1 + \mathscr{I}_2)\psi = \mathscr{I}\psi = 0 \tag{1.2.13}$$

It is important to note, at this stage, that while *the operation* of the interaction coupling term $\mathscr{I}$ on ψ yields a zero value, the coupling functional $\mathscr{I}$ itself does not vanish! This is a consequence of the nonlinear features of the coupled matter field equations for the particle–antiparticle pair—under the special set of circumstances that have been specified here.

With the result (1.2.13) in the field equations (1.1.1), it follows that (1.2.1) is indeed *an exact solution* of the nonlinear field equations for the special case when they describe a coupled particle–antiparticle pair in this particular state, corresponding to each field component in the same state of motion.

1.3. *The Energy and Momentum Associated with Solution* (1.2.1)

It follows from Noether's theorem [see, for example, Bogoliubov & Shirkov (1959)] that the covariance of the field theory with respect to infinitesimal translations in time and space imply, respectively, the conservation of energy, P_0, and momentum, P_k, defined as follows:

$$P_0 = \int \left(\sum_{i=1}^{n} \left[\frac{\partial \mathscr{L}}{\partial(\partial_0 \Lambda_\zeta^{(i)})} \partial_0 \Lambda_\zeta^{(i)} \right] - \mathscr{L} \right) d\mathbf{r} \tag{1.3.1a}$$

$$P_k = \int \sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial(\partial_0 \Lambda_\zeta^{(i)})} (\partial_k \Lambda_\zeta^{(i)})\, d\mathbf{r} \tag{1.3.1b}$$

($k = 1, 2, 3$).

The summations in these integrals are taken over all of the fields $\{\Lambda_\zeta^{(i)}\}$ (ζ denoting the field components), and their conjugates, that describe the system under study. In the case of the bound electron–positron pair, these are the twelve field variables

$$\{\psi^{(e^-)}, \bar{\psi}^{(e^-)}, \psi^{(e^+)}, \bar{\psi}^{(e^+)}, \varphi_\alpha^{(e^-)}, \varphi_\alpha^{(e^-)\dagger}, \varphi_\alpha^{(e^+)}, \varphi_\alpha^{(e^+)\dagger}\}$$

where $\alpha = 1, 2$.

Taking the Lagrangian density $\mathscr{L}$ in equation (1.3.1) to be the sum of the integrands in the contributions to the action functionals in Part III, equations (3.1), (3.3) and (3.4), and inserting the exact solution (1.2.1) for $\psi^{(e^\pm)}$ and the solution (1.2.6) for the electromagnetic field solutions $\varphi_\alpha^{(e^\pm)}$, ($e^+ = -e^- = e$), the substitutions into equations (1.3.1) then gives the result:

$$P_0 = P_1 = P_2 = P_3 = 0 \tag{1.3.2}$$

Thus we see that the particular *exact solution* (1.2.1) of the coupled equations for the particle–antiparticle pair, corresponds to a conserved energy-momentum that is, numerically, a null vector, with each of the four components being identically equal to zero. Because of the latter feature, that all four components of P_ν are zero in some Lorentz frame, they must all be zero in any Lorentz frame—since P_ν is a four-vector. That is to say, the result derived for the *ground state* of the bound electron–positron pair (the state of minimum energy) is Lorentz-invariant.

The reason that the zero energy-momentum, for the problem under study, is the ground state, is due to a feature of any 'classical' field formalism—not allowing *both* positive and negative energy values to simultaneously be in the energy spectrum for the physical system. This follows from the 'classical' definition of energy in terms of a set of generators of *continuous transformations* in the function space that describes the physical system. The transition from positive to negative energy states (or vice versa) requires a discontinuous jump—an allowed type of transition in the quantum theory, but not so in a 'classical' field theory. Thus, starting from zero energy, all other energy values can be either positive or negative—not both (see, e.g. Dirac, 1947, p. 272).

It also follows from the general form of the Lagrangian density used above that its invariance under rotations in space implies, according to Noether's theorem, a form for the conserved angular momentum for the electron–positron pair (in this particular state) that is identically zero (in all of its tensor components, separately). Thus, when the pair is in its ground state of null energy-momentum, this corresponds, spectroscopically, to the $^1S_0$ state.

Note that the features of a bound particle–antiparticle pair that have been predicted thus far are independent of the numerical value for the inertial mass parameter λ. The exact solution (1.2.1) for the ground state of the pair then applies to the proton–antiproton pair (and any other fermion–antifermion pair) as well as the electron–positron pair. This degeneracy—which results from a duplication of the ground state of matter for a whole spectrum of mass values, is, in fact, a consequence of our use of the special relativistic approximation for a formalism that is intrinsically generally covariant. Indeed, it was shown in Part II how the combination of the metrical field equations (in a Riemannian space), with the matter and electromagnetic field equations, leads to a unique discrimination between inertial mass parameters as a local feature of the unification of these field theories. Thus, such degeneracy in the ground state of matter is removed when the curvature of space-time is taken into account.

1.4. *The 'Free Particle' Limit and 'Pair Annihilation'*

When the coupling functional $\mathscr{I}$ in equation (1.1.1) approaches zero, these coupled equations then approach the form of the 'free field' equations for the electron and positron, separately. The solutions are the plane waves which follow from the free particle Dirac theory. With these solutions in the Lagrangian for the system, the conserved energy-momentum for the two uncoupled matter fields is just the four-vector

$$p_\mu^{(e^-)} + p_\mu^{(e^+)} = \{\mathbf{p}^{(e^-)} + \mathbf{p}^{(e^+)}; 2\lambda\} \tag{1.4.1}$$

where $\mathbf{p}^{(e^-)}$ and $\mathbf{p}^{(e^+)}$ are the continuum values of the momenta associated with the limiting 'free particle' and 'free antiparticle' matter fields. The conserved energy, 2λ, and the momenta, follow from the general expressions (1.3.1) when the plane wave solutions of the free particle Dirac equation are inserted (Dirac, 1947, Ch. XI).

To sum up, we have seen that the *general form* of the matter field equations for an electron–positron pair, according to the elementary interaction field theory, predicts two different solutions that correspond to the extrema of the energy spectrum for this two-body system. The *ground state* of null energy-momentum, corresponding to the *exact solution* (1.2.1) of the coupled nonlinear equations (1.1.1), represents the *maximum binding* for the pair. The asymptotic 'free particle' solutions of equation (1.1.1), corresponding, physically, to the interaction functional being arbitrarily close to zero (to describe the *appearance* of two practically free particles—particles that are effectively an infinite distance apart) represents the limit

of *no binding*. Note, once again, that this limit is not physically realizable in practice, or in principle (within the elementary interaction field theory), even though it can be approached arbitrarily closely.

Consequently, this theory predicts that in any given 'rest frame', the range of energy that is available to the particle–antiparticle pair is just equal to

$$\Delta E = (2\lambda - 0) = 2\lambda$$

If we are talking about the electron–positron pair, this is around 1 MeV; if it is a nucleon–antinucleon pair, this is ~2 GeV. The physical meaning of this result is that it should take such quantities of energy transfer to a pair in this ground state (from some outside source) in order to excite it into the state in which the constituent components of the system look as though they are free particles. These events, of course, are observed and they are interpreted as 'pair creation'. Similarly, when a pair of such particle fields (that appear to be free) are in a state such that they are capable of giving up energy to their surroundings, then the maximum energy that can be given up is 2λ. This occurs when the pair goes into its ground state of null energy-momentum. These events are also observed, and interpreted as 'pair annihilation'.

However, this theory *predicts* these effects, that are observed experimentally, do not entail any actual creation or annihilation of matter. The result follows here from the *exact solutions* of a nonlinear, *deterministic* field theory, and the correlation between these solutions and the relative state of binding energy of the *closed* two-component system of particle and antiparticle. These processes, then, are not intrinsically statistical, as it is asserted in quantum field theory. There is no need to invent a 'mechanism' that really 'annihilates' and 'creates' matter since the experimental observations are described here without altering the actual quantity of matter that makes up a closed system.

1.5. *The Continuity of Energy Values*

The general expression for the energy and momentum associated with the field description is given in equation (1.3.1). Since the Lagrangian $\mathscr{L}$ is a function that is continuous with respect to continuous changes of the parameters that appear in its argument, and since the field variables that appear in this functional are continuously distributed in their function space (and are not, generally, solutions of eigenfunction equations) it follows that the values for energy and momentum, according to equation (1.3.1), are *continuously distributed* from 0 to 2λ in the case of the particle–antiparticle pair.

On the other hand, we have seen earlier (Part III) that as the coupling between these matter fields tends towards zero, the field equations (1.1.1) do *approach* the linear eigenfunction form of the quantum theory. In this limit, then, the interaction weighting, which in this theory relates to the observed energy values, approaches a *peaked distribution*. For example, the observed

^3S and ^1S states of 'positronium', that are conventionally interpreted as its bound states, do indicate the peaked distribution for the values of energy and angular momentum when these fields are very weakly coupled, compared with their maximum binding (in these states, the binding is of the order of electron-volts, compared with the maximum binding of the order of a million electron-volts). Even so, the observed energy levels for the ^3S and ^1S states of positronium do have a *finite width*. According to the present theory, the source of the finite width is the nonlinear coupling between the electron and positron fields. Since this can never really 'turn off', the *actual limit* of discrete energy values does not exist, even though it can be approached arbitrarily closely.

The prediction then follows from this theory that as the particle–anti-particle energy spectrum is viewed from minimum binding (i.e. maximum energy $= 2\lambda$) to maximum binding (i.e. minimum energy $= 0$), the peaks in the energy spectrum become less and less sharp, until they eventually wash out altogether, when the relative energy of coupling is sufficiently great.

1.6. *Correlation with the Experimental Facts and the Omission of Photons*

From the experimental data that imply that 2λ units of energy are transferred to an apparatus when a pair 'annihilates', it is also concluded that this energy is distributed equally between two 'photons', $\gamma(\pm k)$, that are emitted simultaneously and propagate in opposite directions. The conventional interpretation then asserts that there are photons in existence at times when matter does not exist. This is in contradiction with the conclusion of the elementary theory that 'photons' need not exist as fundamental entities (Part II). Therefore it is incumbent on this theory to explain all of the experimental facts having to do with 'pair annihilation', that are usually understood in terms of 'photons', without the need to introduce photons at all.

First, it is clear that one does not directly 'see' the photons $\gamma(\pm k)$—they are rather *inferred* from the observed response of the charged matter in a detector (e.g. a Geiger counter) to the other charged matter in the 'source' of interaction. Might it then not be possible that the actual observations can be explained in terms of a direct current–current coupling, without the need to introduce the intermediate 'photons'? It will now be shown that this is indeed the case.

We have seen above that when an electron–positron pair should go into its ground state of null energy-momentum, the maximum energy that would be transferred to a detector—say two Geiger counters, is 2λ. This is in agreement with the experimental facts. To complete the comparison with the experimental facts, however, it is further required to show that when these two Geiger counters are equidistant, along a common axis on each side of a source of electrons and positrons, they would respond simultaneously (i.e. in coincidence), each absorbing the quantity of energy equal to λ.

The simultaneity of the response of the two counters in this problem

follows automatically from the theory since here one does not have separate space-time coordinate systems. The fields describing the interaction between the pair and the detecting apparatus are all mapped in one space-time; that is to say, the interactions that are described by all of the field variables of the theory depend on only one time parameter. Further, since the solution (1.2.1) (in its particular Lorentz frame) does not single out any spatial orientation, it follows from the isotropy of the description, and the feature of this theory that energy is not transferred into free space, that in the macroscopic measurement, where the two Geiger counters actually respond to a large system of particle–antiparticle pairs, as they go into their null energy-momentum state, each counter should absorb half of the total transferred energy. Thus the theory predicts, in agreement with experiment, that there should be a correlation between the 'pair annihilation' process and the *coincident* transferral of λ units of energy (on the average) to each of the two detecting counters.

1.7. *Dynamical Properties of the Ground State* (Sachs, 1968c)

With the idea that the electromagnetic source fields serve only as factors in the description of the coupling of the pair to a detecting apparatus, it might be more instructive at this point to express these variables in terms of the usual charge and current densities of the standard Maxwell theory—as the directly detectable variables of the electron–positron pair.

The proper identification was shown in Part II to have the following form (in a particular Lorentz frame):

$$\begin{aligned} e^{\pm}\bar{\psi}^{(e^{\pm})}\Gamma_1\psi^{(e^{\pm})} &= 4\pi i\begin{pmatrix} -\rho + j_3 \\ j_1 + ij_2 \end{pmatrix} \\ e^{\pm}\bar{\psi}^{(e^{\pm})}\Gamma_2\psi^{(e^{\pm})} &= 4\pi i\begin{pmatrix} -(j_1 - ij_2) \\ \rho + j_3 \end{pmatrix} \end{aligned} \tag{1.7.1}$$

Comparing equation (1.2.2) with equation (1.7.1), the source fields (for $\alpha = 1, 2$) for the pair in its ground state can be expressed with the usual variables as follows:

$$\alpha = 1: \quad \rho = 2e^{\pm}, \quad j_3 = 0, \quad j_1 + ij_2 = 2e^{\pm}\exp(2i\lambda t) \tag{1.7.2a}$$

$$\alpha = 2: \quad \rho = 2e^{\pm}, \quad j_3 = 0, \quad j_1 - ij_2 = 2e^{\pm}\exp(-2i\lambda t) \tag{1.7.2b}$$

It is clear from this form that the ground state of the pair corresponds to two oppositely polarized currents that are mutually transverse with respect to the x_3-direction. Thus, the coincident response of two equidistant counters at the time t would be to two spatially transverse currents that are 90° out of phase with each other. The response of the counters here would be due to the *direct coupling* with these two *distinguishable currents*. This assertion will now be demonstrated.

To derive the effect of the currents $j_1 \pm ij_2 = j_{\pm}$ of the pair (located, say, at the origin) on two counters, at $\pm r$, it will be necessary to calculate the corresponding electric field intensities, $E_{\pm}$, at the locations of the counters.

This, of course, is because it is the electric field intensity that determines the motion of a test charge in the detecting apparatus.

In the Lorentz frame of the detecting device, the electromagnetic vector that corresponds to the oppositely polarized current densities, $j_{\pm}$, is determined from the particular solutions of D'Alembert's equation

$$\Box A_{\pm}(\mathbf{r}', t') = 4\pi j_{\pm}(\mathbf{r}', t') \tag{1.7.3}$$

where $(\mathbf{r}', t')$ are the coordinates of the test charge in the apparatus, while $(\mathbf{r} = 0, t)$ are the coordinates of the source—the pair itself. The solution of equation (1.7.3) is

$$A_{\pm}(r' t') = \int j_{\pm}(t)\, S(x - x')\, d^4 x \tag{1.7.4}$$

where $S(x - x')$ is the Green's function [Part III, equation (3.6)]. This is the form that is symmetric in the advanced and retarded terms. However, it should be noted that because of the symmetry of the experimental set-up in this particular problem, the result to be derived on the responses of the two counters (on opposite sides, along a common axis with, and equidistant from the plane of polarization of the source currents $j_{\pm}$) is insensitive to the appearance or lack of appearance of the advanced term in the Green's function.

Since $j_3 = 0$, the substitution of the Green's function [Part III, equation (3.6)] into equation (1.7.4) (with $r = 0$) yields the following solution:

$$A_3 = \int j_3(t)\, S(x - x')\, d^4 x = 0$$

$$A_{\pm}(r', t') = \frac{(2e^{\pm})}{2r'} \{\exp[\pm 2i\lambda(t' + r')] + \exp[\pm 2i\lambda(t' - r')]\}\, \hat{\mathbf{e}}_{\pm} \tag{1.7.5}$$

where $\hat{\mathbf{e}}_{\pm} = \hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_i$ is a unit vector in the ith direction.

It then follows from (1.7.5) that the electric field intensities $E_{\pm}(r', t')$ at the sites of the counters (representing the effects of the polarized current densities $j_{\pm}$ of the particle–antiparticle source) have the form:

$$E_{\pm}(r', t') = -\frac{\partial A_{\pm}}{\partial t'} = \mp \frac{(2i\lambda e^{\pm})}{r'} \{\exp[\pm 2i\lambda(t' + r')] + \exp[\pm 2i\lambda(t' - r')]\}\, \hat{\mathbf{e}}_{\pm} \tag{1.7.6}$$

Thus we see that $E_{\pm}$ describes the wave motion of an oscillating charge, with angular frequency $\omega = 2\lambda = 2mc^2/\hbar$ and a propagation vector whose magnitude is ω/c. (m is the rest mass of the electron. Planck's constant $h = 2\pi\hbar$ and the speed of light, c, are inserted above for illustrative purposes. We are using units throughout with $\hbar = c = 1$.)

It follows, then, that when the phase of the current has some fixed value, say zero, at $t = 0$, then the phase of the electric field at r', which is produced by this current, would not become zero until the later time $t' = r'/c$ in the retarded solution, and $t' = -r'/c$ in the advanced solution. Thus, the *magnitude* of the time taken for propagation of the electromagnetic inter-

action between the pair and each of the detectors (that are located a distance r' from the pair) is r'/c.

A salient point is that for the solution E_+, the sign of the propagation vector is positive in the retarded term and the sign of this vector is negative in the advanced term. The oppositely polarized current density, j_-, gives rise to the electric field intensity E_- with the same functional form as E_+ except that the propagation vector in this case is negative in the retarded term and positive in the advanced term.

Thus we see that the oppositely polarized currents, $j_\pm$, at the common spatial location ($r = 0$) give rise to oppositely polarized electric field vectors that propagate in opposite directions—such that when each counter is an equal distance r' on each side of the pair, along a common axis, they will simultaneously detect oppositely polarized currents at the time $t' = r'/c$, each absorbing the energy equal to mc^2.

This *derived* result agrees with the experimental facts and with the law of energy conservation. It also agrees with the prediction of the model of quantum field theory which asserts that two photons are simultaneously created when the pair 'annihilates'. However, the present theory does not require the introduction of 'photons' nor that matter should really 'annihilate'. A most important difference in the two theories lies in the *deterministic* field approach of this derivation of the experimental facts, compared with the *intrinsically statistical approach* of the quantum field theoretic assertion about these facts.

1.8. *The Wu–Shaknov Experiment and Angular Momentum Conservation* (Wu & Shaknov, 1950)

According to the experimental observations, the electron–positron pair are in a singlet S-state just prior to the event interpreted as 'annihilation'. This implies that the two photons that are supposedly created in the process must be oppositely polarized in a mutual plane that is transverse to their oppositely directed motion—if angular momentum is to be conserved. The preceding derivation of the elementary interaction theory, relating to the detected pair of currents of the electron–positron system in its ground state, also agrees with this assertion about angular momentum conservation. In addition, however, the quantum mechanical requirement for an anti-symmetric wave function to describe the pair in its singlet state, implies that the final state of two photons must correlate the phases of their polarization vectors with a 90° phase difference (also predicted by the present theory, as we have seen above).

To test the latter consequence, Wu & Shaknov (1950) designed an experiment to measure the coincident counts of two linearly polarized radiation fields that have been Compton scattered through an angle θ, relative to their initial direction of propagation. The significance of this experiment lies in the sensitivity of the Compton cross-section (for the scattering of coincident radiation beams) to the correlation of their polarizations.

When the mechanism of scattering is the Compton effect, and the two radiation fields are perpendicularly polarized, it follows from the Klein–Nishina formula (Heitler, 1944) that the ratio of coincident counts for perpendicularly polarized radiation is exactly 2.00 for cases when the scattering planes (formed by the initial and scattered directions of the radiation) are perpendicular and parallel, and where the angle of scattering in these planes is averaged over. The ratio that was measured by Wu and Shaknov was $2{\cdot}04 \pm 0{\cdot}08$. Consequently, this experiment confirmed the quantum mechanical prediction about a 90° phase correlation in the coincidentally scattered radiation beams. But according to the development in the preceding paragraphs, this result was also an experimental verification of the prediction of the elementary interaction theory.

1.9. *The Compton Effect*

Since the conclusions from the results of the Wu–Shaknov experiment depend on an identification with the usual expression of the Compton cross-section, it is necessary to show here how the same expression appears within the elementary interaction field theory.

Considering the process in which an electron–positron pair is scattered by an electron, this theory actually sees this process in terms of a tightly bound pair (the 'projectile') coupled weakly to a third matter field component (the 'target') *of a closed system.* The mathematical description, then, is in terms of a set of three coupled field equations [of the type in Part III, equation (3.8)] in which one of the equations can be assumed to be approximately uncoupled. In this approximation, the latter takes the form of the Dirac equation for a 'free particle'. The pair, in turn, is described as it was in the preceding paragraphs [equation (1.1.1)].

The next step, to derive the Klein–Nishina formula, is to introduce a small coupling between the pair and the target (almost free) electron and to treat this as a perturbation on the free particle solutions for the target electron.

We have seen that when the particle–antiparticle pair is in its ground state of null energy-momentum, it behaves, dynamically, as a pair of oppositely polarized current densities, whose phases are correlated with a 90° difference; and that in this state, the pair has the same dynamical features as a pair of photons that are conventionally evoked to explain the data on 'pair annihilation'. The electromagnetic potential that corresponds to the independent currents (1.7.2), and solves equation (1.2.7), has the same time behavior as the current densities. If one should make a Lorentz transformation to the rest frame of the (assumed uncoupled) target electron, then the effective vector potential for the pair that acts on the target electron (rather than on an apparatus, as in the preceding example) takes the following form, in a unit volume:

$$A_3 = 0, \qquad A_{\pm} \propto \exp\left[\pm i(\omega t - \mathbf{k}\cdot\mathbf{r})\right] \tag{1.9.1}$$

where

$$\omega = \frac{2mc^2}{\hbar}[(1-\beta)/(1+\beta)]^{1/2}, \qquad \mathbf{k} = \frac{\omega}{c}\frac{\boldsymbol{\beta}}{\beta}$$

and β is the relative velocity of the target and projectile systems (in units of c).

With the exact solution (1.2.1) for the pair in its ground state, and the approximation of weak coupling between the pair and the target electron, the usual formal perturbation technique can be applied to determine the scattering cross-section. To first order, the perturbing interaction is $e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$, where ψ is the free field Dirac solution for the target electron and A_{μ} is the effective electromagnetic four-potential—the space part given in equation (1.9.1). The Coulomb component, A_0, *for the electron–positron pair* is, of course, zero.

The interaction, $e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$, between the particle–antiparticle pair (as the 'projectile') and the target electron, which is predicted by the elementary interaction theory, is then seen to be identical with the interaction between a projectile 'photon' and the target electron, in the Compton effect. It then follows that the application of perturbation theory to this problem yields the Klein–Nishina formula for the Compton cross-section, derived in the usual way. The predictions of this theory are then identical with those of the quantum theory in regard to the correlation of the polarizations that was observed in the Wu–Shaknov experiment.

It should be noted at this stage, however, that if the coupling between the pair and the target electron should become increasingly great, the tendency would be to excite the pair into a state that no longer displays the dynamical properties that are identical with those of 'photons'. It rather might indicate a behavior that appears as 'pair creation'. To describe the latter process in a quantitative fashion would then require the consideration of the solutions of three coupled nonlinear equations—for the projectile 'pair' and the target electron—all strongly coupled.

1.10. *Blackbody Radiation* (Sachs, 1965)

It will now be shown that the experimental results that are used to deduce the properties of a 'photon' gas from the spectral distribution of blackbody radiation are equally explained in terms of an *ideal gas* of particle–antiparticle pairs, rather than photons, when the pairs can be assumed to be in their ground states of null energy momentum, as derived above.

A cavity is maintained at a constant temperature so that the 'radiation' within it can come to thermodynamic equilibrium with the matter of the cavity walls. The energy density within the cavity is then measured as a function of frequency from the response of charged matter (the detecting apparatus) that couples to the inside of the cavity through a small window in its wall.

The first remarkable feature of these experiments was the insensitivity of the resulting spectral distribution to the nature of the constituent material

(i.e. the atomic makeup) of the cavity walls. It then appeared that the radiation in the cavity was not dynamically coupled to the states of motion of charged matter that constitute the walls—but is rather an entity that, once emitted by the walls, is independent of this matter. Then, it was found that when kept in thermodynamic equilibrium with the cavity walls, at the constant temperature T, the spectral distribution of this radiation could be fit to the Planck formula:

$$du/d\nu = \frac{(8\pi h\nu^3)}{c^3}\frac{1}{\exp(h\nu/kT)-1} \tag{1.10.1}$$

where du is the density of radiant energy in the frequency range between ν and $\nu + d\nu$ at the temperature T. This behavior of the energy density with frequency and temperature, as well as the experimental fact about its insensitivity to the type of material that constitutes the cavity, are the empirical facts about blackbody radiation that are to be explained.

Instead of a 'radiation gas', let us assume that the cavity in this experimental set-up is populated by an *ideal gas* of particle–antiparticle pairs that are each in their ground states of null energy-momentum. We will now express the full set of coupled field equations [Part III, equation (3.8)] as follows:

$$\begin{aligned}
&0(1^+,1^-,2^+,2^-,\ldots,n^+,n^-;f)\psi^{(1^+)}(x)=0\\
&0(1^-,1^+,2^+,2^-,\ldots,n^+,n^-;f)\psi^{(1^-)}(x)=0\\
&\qquad\qquad\vdots\\
&0(n^-,n^+,1^+,1^-,\ldots,(n-1)^+,(n-1)^-;f)\psi^{(n^-)}(x)=0\\
&0(f;1^+,1^-,\ldots,n^+,n^-)\psi^{(f)}(x)=0
\end{aligned} \tag{1.10.2}$$

where

$$0(1^+,1^-,\ldots,n^+,n^-;f)=\left\{\gamma_\nu\partial^\nu-\lambda-\mathscr{I}(e^{1^-})+\sum_{k\neq 1^-,1^+}\mathscr{I}(k)\right\}$$

The labels $n^\pm$ in these equations refer to the particular $(e^- - e^+)$ set in the gas that are tightly coupled. The two solutions, $\psi^{(n^+)}$ and $\psi^{(n^-)}$ that are denoted in this way are related according to the charge conjugation transformation [equation (1.1.1)]. The labels f refer to the field variables that relate to the interacting matter fields of the cavity walls and the detecting apparatus.

In the asymptotic limit, when the term $\mathscr{I}(k)$, $(k \neq 1^+, 1^-)$ can be assumed to have a negligibly small effect on the solution $\psi^{(1^+)}$ compared to the effect of $\mathscr{I}(1^-)$ on this solution, the field equations (1.10.2) reduce to $n/2$ uncoupled sets of two coupled field equations for the particle–antiparticle pairs (as well as the field equations that describe the interacting charges in the cavity walls and the detecting apparatus). In the consideration of this asymptotic limit, note that the formalism reduces to the description of $n/2$ *distinguishable* interactions.

It was shown earlier, from the features of the ground state solution (1.2.1) of the pair, that the sum of the electromagnetic equations for the electron

and positron field, when each are in the same state of motion, gives the equation (1.1.4) that is source-free. Recalling that the physically admissible solutions of these equations are only the particular solutions (according to the elementary interaction theory), the electromagnetic field intensities $\varphi_\alpha^{(\text{pair})}$, associated with the pair, *as a unit*, is identically zero. It then follows that the ideal gas of such pairs in the cavity will not couple electromagnetically to the walls—they will appear as though they are dynamically uncoupled from the moving charged matter in the walls of the cavity, a prediction that is in agreement with the experimental facts.

Recall also, however, that each of the two components for the pair can separately couple, electromagnetically, to other charged matter. Thus, when in the ground state of null energy-momentum, each of the components of the pair is capable of absorbing some energy from the walls of the cavity—from its direct electromagnetic coupling to currents in the wall (which would occur in an incoherent fashion)—thereby decreasing the relative binding of the pair to *any* amount. (Recall that the relative energies of the pair are in a continuum, rather than a discrete spectrum.) The pair would eventually convert this absorbed energy into increased kinetic energy, before giving the energy back to the charged matter in the walls, thereby returning to its ground state of null energy-momentum. Such is the energy exchange process that maintains this ideal gas of pairs at a constant temperature T, with respect to the walls and bath that are maintained at this temperature.

The energy density that is detected in the cavity is predicted from the expressions of the form given in equation (1.3.1a), where the sum is now taken over all electron and positron solutions for the ideal gas of pairs, the cavity walls, and the detecting apparatus. With the energy of mutual coupling for each of the pairs being zero (when in the ground state, described by the solution (1.2.1)) the only nonzero contribution to (1.3.1a) comes from the coupling of the pairs, in terms of their individual electron and positron components, to the charged matter of the detecting apparatus. The corresponding *matter field components* are the stationary states with frequency ω_0. The latter frequencies are the controlled values in the experimental arrangement of this study (e.g. the frequencies associated with the light filters, etc.). Thus the fields that contribute to the observed energy density have the form

$$\psi^{(0)} \propto V^{-1/2} \exp(-i\omega_0 t)$$

It follows that the corresponding energy density contributions reduce to the form

$$P^{(\omega_0)} = \int \theta_{00}\, d\mathbf{r} = \int i\hbar\psi^{(0)\dagger}\, \partial t \psi^{(0)}\, d\mathbf{r} = \hbar\omega_0 \tag{1.10.3}$$

for each frequency ω_0 of the detecting apparatus.

Using the effective vector potential associated with each of the pairs in the cavity [equation (1.9.1)] and the continuous distribution of frequencies relative to a given frame of reference [indicated in equation (1.9.1)] it follows that for any detector frequency ω_0, the apparatus will respond to

the interior of the cavity (through a small window in its wall) at frequencies that are integral multiples of this driving frequency. This follows from the finiteness of the cavity and the requirement of restricting the interaction to its domain. For a given detector frequency ω_0, the energy of the mth mode of interaction between the detector and the ideal gas of pairs (in a volume V) is then

$$(V\theta_{00})_m = m\hbar\omega_0 \qquad (m = 1, 2, 3, \ldots) \tag{1.10.4}$$

We see here that the response of the detecting apparatus to the individual pairs in the cavity will display a set of *distinguishable interactions*, each labelled with the particular mode of oscillation m.

With the establishment of a constant temperature, T, in the gas of pairs, Maxwell–Boltzmann statistics may then be applied to the system of *distinguishable interactions*, to determine the statistically averaged energy. Thus,

$$V\langle\theta_{00}\rangle = \frac{\sum_{m=0}^{\infty} \exp[-m\hbar\omega_0/kT](m\hbar\omega_0)}{\sum_{m=0}^{\infty} \exp[-m\hbar\omega_0/kT]} = \frac{\hbar\omega_0}{\exp(\hbar\omega_0/kT) - 1} \tag{1.10.5}$$

Finally, dividing this expression by the volume of the cavity and then weighting the averaged energy density with the differential increment of detecting frequency modes per frequency interval, we arrive at the expression for the energy density per frequency interval—the quantity that is to be compared with the observations.

The density of modes is derived in the usual way from the volume of wave vector space, as follows:

$$\frac{dg}{d\omega_0} = \frac{d}{d\omega_0}\left[\frac{2}{2\pi^3}\int d\mathbf{k}_0\right] = \frac{\omega_0^2}{\pi^2 c^3} \tag{1.10.6}$$

The factor 2 appears in equation (1.10.6) because of the fact that the detector, with a given frequency, can respond to either of the two source fields [equation (1.7.1)] that are associated with the pair in the cavity. These, in turn, correspond to the two oppositely polarized currents.

The product of the right-hand sides of equations (1.10.5) and (1.10.6) is the density of interaction energy per interval of the detectable frequency spectrum. The result

$$\frac{du}{d\omega_0} = \langle\theta_{00}\rangle\frac{dg}{d\omega_0} = \frac{\hbar\omega_0^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\omega_0/kT) - 1} \tag{1.10.7}$$

is the desired distribution function that was originally discovered by Planck and shown to agree with the spectral distribution of blackbody radiation.

With this result and the preceding analysis, which indicated that the observations interpreted as pair annihilation and creation can be derived from the features of a bound state of a particle–antiparticle pair—without the need to introduce photons at all—it has been demonstrated that

electrodynamics, within the completely field theoretic approach to an action-at-a-distance theory (the elementary interaction theory) can indeed totally dispense with the photon concept.

2. *The Electron–Proton System*

Our determination of the exact solution (1.2.1) for the ground state of the bound electron–positron pair was facilitated by the intrinsic symmetry in the field equations for this particular case. Specifically, the symmetric features of the problem that led to the solution were (1) the inertial mass parameter for each of the coupled fields are the same and (2) the electron and positron solutions each corresponded to the same state of motion.

In this section, we treat a two-body system that has neither of these symmetry properties. The mass parameters, m and M, for the electron and proton respectively, differ by several orders of magnitude. Further, the states of motion of these two components of the system are quite different—it will be assumed that the proton is stationary while the electron field corresponds to an orbiting motion.

At the present stage of the investigation no exact solution has been found for the electron–proton system. The solutions that will be studied here instead follow from an approximation scheme in which the field equations are *linearized.* This is justified in view of the fact that the ratio of masses, m/M, is sufficiently small so that the actual momentum transfer to the proton field *in this bound system* can be assumed to be negligible. This is physically equivalent to the statement that for the binding energies considered in the bound states of hydrogenic atoms, one can neglect the recoil of the proton. Thus we assume that the proton field describes a source of binding at a point, that is, it is at a stationary location, relative to the coordinates of the electron field. This, of course, is equivalent to using Dirac's assumption, in his structuring of the hydrogen equation, when he takes the *covariant* term $\bar{\psi}^{(e)}\gamma_\mu\psi^{(e)}A_\mu^{(p)}$ and approximates it by the Coulomb term $\psi^{(e)\dagger}\psi^{(e)}A_0^{(p)}$, where $A_0^{(p)} = ie/r$ and $r = 0$ locates the stationary proton.

With this approximation, it will be shown that the entire hydrogen spectrum, *including the Lamb shift*, emerges from the coupled field equations for the electron–proton system. It will then be shown how the enlargement of this two-particle system that would include a background (ideal) gas of particle–antiparticle pairs, in their ground states of null energy-momentum (derived in the preceding section) leads to the correct lifetimes for the excited states of hydrogen.

2.1. *Linearization of the Field Equations* (Sachs & Schwebel, 1961)

Starting with the field equations [Part III, equations (3.8) and (3.9)], applied to the electron–proton system, we must consider the solutions of the coupled equations:

$$(\gamma_\mu\partial^\mu - \mathscr{I}(p) + \lambda)\psi^{(e)} = 0 \qquad (2.1.1a)$$

$$(\gamma_\mu\partial^\mu + \mathscr{I}(e) + \Lambda)\psi^{(p)} = 0 \qquad (2.1.1b)$$

where λ and Λ are the reciprocals of the Compton wavelengths, $mc/\hbar$, $Mc/\hbar$, of the electron and proton, respectively, and m and M are their inertial masses. The coupling field, $\mathscr{I}$, is defined for electrodynamics in Part III, equation (3.9).

We will now consider the proton solution, $\psi^{(p)}$, to have the following stationary form:

$$\psi^{(p)} = \exp(-i\Lambda t) f(\mathbf{r}) s \tag{2.1.2}$$

where s is a *constant four-component spinor*, the only requirement on it, generally, being that it must be normalized, i.e. $s^+ s = 1$.

Dividing equation (2.1.1b) by Λ, we have the equivalent equation:

$$\left(\frac{1}{\Lambda}\gamma_0 \partial^0 - \frac{1}{\Lambda}\gamma_k \partial^k + \frac{e}{\Lambda}\mathscr{I}(e) + 1\right)\psi^{(p)} = 0 \tag{2.1.3}$$

The operator, $(1/\Lambda)\gamma_k \partial^k$, corresponds to the ratio of kinetic energy of the proton to its rest energy. Since we have assumed that the proton's motion (relative to the electron)—i.e. its recoil energy—is negligible compared with its rest energy (~1 GeV), we may neglect this term. The operator, $e\mathscr{I}(e)$, corresponds to the ratio of electron–proton binding (~eV) to the proton's rest energy. This may also be assumed small enough to neglect (for the purposes at hand). Neglecting these two terms, and choosing $s^+ = (1\ 0\ 0\ 0)$ (in this Lorentz frame), equation (2.1.3) approaches the form:

$$\left(\frac{1}{\Lambda}\gamma_0 \partial^0 + 1\right)\exp(-i\Lambda t) f(\mathbf{r}) s = 0 \tag{2.1.4}$$

That is, with these approximations, the function (2.1.2) correspondingly approaches an exact solution of the proton equation (2.1.1b).

Next, it is observed that the space-dependent part, $f(\mathbf{r})$, of $\psi^{(p)}$ may be chosen arbitrarily, without altering the validity of equation (2.1.4). Appealing once again to the physical argument that led to the assumption of a stationary point proton, we may take $f(\mathbf{r})$ to be defined such that

$$|f(\mathbf{r})|^2 = \delta(\mathbf{r}) \tag{2.1.5}$$

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function.

The insertion of the resulting proton solution into $\mathscr{I}(p)$ in the electron equation (2.1.1a) then gives rise to the explicit form of the operator that determines the electron states in hydrogenic atoms.

Using equation (2.1.5) in equation (2.1.2),

$$\bar{\psi}^{(p)} \gamma_k \psi^{(p)} = 0, \qquad \bar{\psi}^{(p)} \gamma_0 \psi^{(p)} = \delta(\mathbf{r}) \tag{2.1.6}$$

where $k = 1, 2, 3$. According to Part III, equation (3.9), the interaction coupling term $\mathscr{I}(p)$ has two parts. Inserting (2.1.6) into the first part, we have

$$\mathscr{I}(p)_1 = \gamma_0 e/r \tag{2.1.7}$$

This is the term that arises from the Coulomb potential of the proton, at the electron site, in the usual Dirac theory for hydrogen.

Before the second part, $\mathscr{I}(p)_2$, can be determined, we must know the explicit form of the spinor solutions $\varphi_\alpha^{(p)}$ of the electromagnetic field equations. Inserting the proton solutions into the electromagnetic field equations, Part II, equation (1.5), we have

$$\begin{aligned}\sigma_\mu \partial^\mu \varphi_1^{(p)} &= -4\pi i e \delta(\mathbf{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sigma_\mu \partial^\mu \varphi_2^{(p)} &= 4\pi i e \delta(\mathbf{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned} \tag{2.1.8}$$

The solutions of these equations have been determined (Sachs, 1971a) where it was found that

$$\begin{aligned}\varphi_1 &= -\frac{ie}{r^3} \begin{pmatrix} x_3 \\ x_1 + ix_2 \end{pmatrix} \\ \varphi_2 &= \frac{ie}{r^3} \begin{pmatrix} x_1 - ix_2 \\ -x_3 \end{pmatrix}\end{aligned} \tag{2.1.9}$$

Finally, substituting the solutions (2.1.9) into $\mathscr{I}(p)_2$ and using the result $a_1 = -a_2 = +1$ (that was determined in the previous analysis of the ground state of the particle–antiparticle pair) the following explicit expression is obtained:

$$\mathscr{I}(p)_2 = 16\pi(ig_M)\frac{e}{r^3}(\mathbf{r} \times \boldsymbol{\gamma})_3 \tag{2.1.10}$$

The noncovariant 'look' of this interaction term is a consequence of using the approximation in which the vector potential A_μ in the electron equation is replaced by the single term A_0 (for the Coulomb potential). Note, however, that the *exact* form for this term (the second part of the functional in Part III, equation (3.9)), is clearly relativistically covariant. This (exact) form is a form of one of the relativistic invariants of the theory, denoted earlier by combinations of $\varphi_\alpha^\dagger Y_\alpha$ (Part II). These are invariants of the spinor form of electromagnetism that we have indicated earlier do not have any counterpart in the standard vector representation of the theory.

Combining equations (2.1.7 and (2.1.10), the *linearized* field equation (2.1.1a) for hydrogen can be expressed in the form:

$$\left\{-\alpha_r p_r - i\frac{\alpha_r}{r}\beta\hat{K} + \frac{e^2}{r} + (16\pi g_M e^2)\frac{(\mathbf{r} \times \boldsymbol{\alpha})_3}{r^3} - \beta\lambda + E\right\}\psi^{(e)} = 0 \tag{2.1.11}$$

The following notation has been used above:

$$\beta = \gamma_0, \qquad \boldsymbol{\alpha} = i\gamma_0\boldsymbol{\gamma}, \qquad \hat{K} = \beta(\boldsymbol{\sigma}\cdot\mathbf{L} + 1), \qquad p_r = -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \quad \alpha_r = \frac{\boldsymbol{\alpha}\cdot\mathbf{r}}{r} \tag{2.1.12}$$

where $\mathbf{L}$ is the orbital angular momentum operator $-i\mathbf{r} \times \boldsymbol{\nabla}$.

Following the usual procedure (Schiff, 1949), we introduce the following change of variables:

$$\rho = \eta r$$

where

$$\eta = +\sqrt{(\eta_1 \eta_2)}, \qquad \eta_1 = \lambda + E, \qquad \eta_2 = \lambda - E$$

The wave equation (2.1.11) then takes the form:

$$\hat{\mathscr{H}}\psi = (\hat{\mathscr{H}}_0 + \hat{V})\psi = -i\frac{E}{\eta}\psi \tag{2.1.13}$$

where

$$i\eta\hat{\mathscr{H}}_0 = i\left\{\alpha_\rho\left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right) - \frac{\alpha_\rho \beta \hat{K}}{\rho} - i\left(\frac{\gamma}{\rho} + \frac{\beta\lambda}{\eta}\right)\right\}\eta \tag{2.1.14}$$

is the unperturbed (generalized) Dirac Hamiltonian for the hydrogenic electron, and γ is the fine structure constant, $e^2/\hbar c$.

If we should now make use of the well-known derivation of the eigenfunctions and eigenvalues of the Dirac unperturbed Hamiltonian for hydrogen, the extra term, $\hat{V}$, in equation (2.1.13) may then be applied as a perturbation on these eigenfunctions, and the corresponding effect of the Dirac eigenvalues may then be determined.

The eigenvalues of the unperturbed Dirac Hamiltonian, $i\hat{\mathscr{H}}_0$ for hydrogen may be expressed as follows:

$$\frac{E_{Jn}}{\eta} = \frac{\lambda}{\eta}\left[1 + \frac{\gamma^2}{(s+n)^2}\right]^{-1/2} \tag{2.1.15}$$

where

$$s^2 = K^2 - \gamma^2, \qquad K^2 = (J + \tfrac{1}{2})^2$$

J is the total angular momentum quantum number and n is any positive integer $(0, 1, 2, \ldots)$. In the usual spectroscopic notation, $(n+1)$ is called the 'principal quantum number'. The energy eigenvalues [equation (2.1.15)] are precisely those which are predicted by the Dirac theory of hydrogen. Except for the Lamb splitting, these are in quite close agreement with the energy level spectrum of hydrogen. That is to say, without the term $\hat{V}$ in the Hamiltonian, (2.1.13), the present theory (with the linear approximation that has been used in this section) is in exact agreement with the Dirac theory of hydrogen.

2.2. *The Lamb Splitting*

The feature of the predicted energy spectrum (2.1.15) that does not agree with the actual data is the *accidental degeneracy* that appears in the Dirac theory, in the hydrogen states whose principal quantum numbers are greater than (their minimum value of) unity. Thus, the Dirac states with the pair of eigenvalues $\pm K$ (of the operator $\hat{K}$) correspond to the same energy values E_{Jn}. When, for example, $J = \frac{1}{2}$, the states $(n+1)S_{1/2}$ and $(n+1)P_{1/2}$ of the Dirac theory are degenerate. Nevertheless, it was discovered by Lamb and his co-workers (Triebwasser *et al.*, 1953; Lamb & Sanders, 1956) that the energy of the $2S_{1/2}$ state is greater than that of the $2P_{1/2}$ state, and

similarly, the $3S_{1/2}$ energy is greater than the $3P_{1/2}$ energy. If this result is to be predicted by the elementary interaction theory, it must come from the additional interaction $\hat{V}$ that appears in the completed Hamiltonian (2.1.13). It will now be shown that this is indeed the case.

The perturbing potential

$$i\hat{V} = \frac{\kappa}{\rho^2}\frac{(\boldsymbol{\rho} \times \boldsymbol{\alpha})_3}{\rho} \tag{2.2.1}$$

in equation (2.1.13) can be seen to lift the accidental degeneracy in the eigenstates of $i\hat{\mathscr{H}}_0$ because of its lack of reflection symmetry in both space and time. The strength of this interaction is measured by the constant

$$\kappa = 16\pi g_M \gamma\eta = \frac{16\pi}{[(s+n)^2+\gamma^2]^{1/2}}\left(\frac{g_M}{\lambda_C}\right)\gamma^2 \tag{2.2.2}$$

where λ_C is the (reduced) Compton wavelength ($\hbar/mc$). This is the reciprocal of the mass parameter that we have denoted as λ.

It is observed that the potential $i\hat{V}$ diverges at the origin as ρ^{-2}. In order to ensure that the solutions of the unperturbed matter field equations have a behavior at the origin that would lead to a rapidly convergent perturbation expansion of functions that depend on ρ^{-n} $(n \geqslant 2)$, let us re-define the unperturbed and perturbing energy operators in equation (2.1.13) by adding and subtracting the term

$$i\kappa\alpha_\rho/\rho^2$$

This expression is not chosen only because it depends on the radial coordinate in the same way as $\hat{V}$; it also depends on the Dirac matrix $\boldsymbol{\alpha}$, which mixes the large and small components of the Dirac (unperturbed) solutions, just as the actual perturbing potential $\hat{V}$ does.

Thus, we take

$$i\hat{\mathscr{H}}_0' = i\left(\hat{\mathscr{H}}_0 - \frac{\kappa\alpha_\rho}{\rho^2}\right) \tag{2.2.3}$$

as the unperturbed Hamiltonian operator and the remaining portion,

$$i\hat{V}' = \frac{\kappa}{\rho^2}\left[\frac{(\boldsymbol{\rho} \times \boldsymbol{\alpha})_3}{\rho} + i\alpha_\rho\right] \tag{2.2.4}$$

may be considered as the perturbing term. Clearly, the problem has not been altered by this change, since

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{V} = \hat{\mathscr{H}}_0' + \hat{V}'$$

The electron equation

$$\hat{\mathscr{H}}_0'\psi = -\frac{i}{\eta}E\psi$$

can be solved *exactly*, giving the solutions

$$\psi = \psi_D \exp(-\kappa/\rho) \tag{2.2.5}$$

where ψ_D are the eigenfunctions of the original Dirac Hamiltonian [equation (2.1.14)] for hydrogen. Note that ψ converges at the origin more rapidly than any polynomial in $1/\rho$ would diverge there.

The accidental degeneracy in the first excited state of hydrogen is described by the two orthogonal wave vectors, $|+K\rangle$ and $|-K\rangle$, that correspond to the same energy value, according to equation (2.1.15). The eigensolutions that are characterized by the angular momentum quantum numbers, $J=\frac{1}{2}, J_3=+\frac{1}{2}$ are as follows: (Bethe & Salpeter, 1957).

$$|-K\rangle_n = \begin{pmatrix} F_-(n) \\ 0 \\ -i\cos\theta G_-(n) \\ -i\sin\theta\exp(i\varphi)\, G_-(n) \end{pmatrix} \exp(-\kappa/\rho) \tag{2.2.6}$$

$$|+K\rangle_n = \begin{pmatrix} \cos\theta F_+(n) \\ \sin\theta\exp(i\varphi)\, F_+(n) \\ -iG_+(n) \\ 0 \end{pmatrix} \exp(-\kappa/\rho) \tag{2.2.7}$$

where $|K|=1.\eta F_\pm/\rho$ and $\eta G_\pm/\rho$ are respectively the large and small components of the radial solutions of the Dirac equation for hydrogen, with the appropriate values of $K=\pm 1$.

It is readily verified that after the integration over angular variables has been carried out, the pertinent matrix elements have the values

$$\langle \pm K|i\hat{V}'|\mp K\rangle = 0 \tag{2.2.8}$$

$$\langle \pm K|i\hat{V}'|\pm K\rangle = \frac{\pm 4\tau}{3\eta}\int_0^\infty \frac{F_\pm G_\pm}{\rho^2}\exp(-2\kappa/\rho)\,d\rho \tag{2.2.9}$$

where (inserting $\hbar$ and c)

$$\tau_n = \eta\hbar c\kappa = \frac{16\pi}{(s+n)^2+\gamma^2}\left(\frac{g_M}{\lambda_C}\right)\gamma^3(mc^2) \tag{2.2.10}$$

It then follows that the difference between the energy values associated with the states, conventionally labelled as $(n+1)S_{1/2}$ and $(n+1)P_{1/2}$, is

$$E[(n+1)S_{1/2}] - E[(n+1)P_{1/2}] \equiv \Delta_n$$
$$= \frac{64\pi}{3[(s+n)^2+\gamma^2]}\left(\frac{g_M}{\lambda_C}\right)\gamma^4|I_+ + I_-|_n(mc^2) \tag{2.2.11}$$

where

$$|I_\pm|_n = (\eta\gamma)^{-1}\int_0^\infty \frac{F_\pm(n)\,G_\pm(n)}{\rho^2}\exp(-2\kappa/\rho)\,d\rho \tag{2.2.12}$$

The integrals in equation (2.2.12) are evaluated in the Appendix in terms of combinations of modified Bessel functions of the second kind (usually

denoted by K_m). Using the explicit expressions for the radial Dirac functions, the resulting integrations (in the Appendix) give the following values for $|I_+ + I_-|_n$, for principal quantum numbers 2, 3 and 4 (to the leading order in γ^2):

$$|I_+ + I_-|_1 = (1/3) + 0(\gamma^2) \tag{2.2.13a}$$

$$|I_+ + I_-|_2 = (2/9) + 0(\gamma^2) \tag{2.2.13b}$$

$$|I_+ + I_-|_3 = (1/6) + 0(\gamma^2) \tag{2.2.13c}$$

Combining equations (2.2.10), (2.2.11) and (2.2.13), it is found that to order γ^4, the Lamb splitting in the first excited state of hydrogen (principal quantum number $(n+1) = 2$) is

$$\Delta_1 = (16\pi/9)\,\gamma^4(g_M/\lambda_C)\,(mc^2) \tag{2.2.14}$$

The ratio of the Lamb splitting in the next excited state, $(n+1) = 3$, to the splitting Δ_1 is then found to be

$$\Delta_2/\Delta_1 = 0{\cdot}2965\ldots \tag{2.2.15}$$

The significance of theoretical *ratio* of Lamb splittings is that, to the accuracy which is required to make the comparison with the data, *its magnitude is independent of the extra fundamental constant of this theory*, g_M. Only after a favorable comparison can be made with the *ratio* in experiment and theory (where there are no adjustable parameters!), will we use the most accurately measured Lamb splitting to determine the magnitude of g_M.

The experimental values for the Lamb splittings are:

$$[E(2S_{1/2}) - E(2P_{1/2})]/h = 1057{\cdot}77 \pm 0{\cdot}01 \text{ MHz}$$

$$[E(3S_{1/2}) - E(3P_{1/2})]/h = 315{\cdot}0 \pm 0{\cdot}3 \text{ MHz}$$

Thus, the experimental ratio to be compared is

$$(\Delta_2/\Delta_1)_{\text{exp}} = (315{\cdot}0 \pm 0{\cdot}3)/(1057{\cdot}77 \pm 0{\cdot}01)$$

$$= 0{\cdot}2978 \pm 0{\cdot}0003 \tag{2.2.15'}$$

The comparison of the theoretical ratio (2.2.15) and the experimental ratio (2.2.15') reveals that their difference is only of the order of 0.2%. In view of the approximations that have been used in this theoretical analysis of the hydrogen spectrum, it can be concluded that at the present stage of computation, the theory is successful in its comparison with the data.

With this result, then, it has been demonstrated that the elementary interaction field theory predicts the observed energy level spectrum of hydrogen—including the Lamb splitting. It should be emphasized that, in contrast with the conventional explanation of the Lamb splitting—which appeals to the enlargement of the electron–proton system so as to include an infinite sea of radiation and pairs, annihilating and being created (at arbitrary times) but on the average causing the proper fluctuations in the spectrum of hydrogen to give the effect, the present theory derived the Lamb effect as a property of the electron–proton system *alone*. It has followed here from the generalization of the Coulomb potential in the hydrogen equation,

which appears as a natural consequence of the factorization of the electromagnetic field equations from the standard (Maxwell) vector representation to the two-component spinor representation. The extra interaction in the Hamiltonian for hydrogen is an approximation for the generalized interaction terms that appear in this theory which have *no counterpart in the standard Maxwell formulation.* In this way, the Coulomb potential, e/r, in the hydrogen wave equation, is generalized by adding a term that is less symmetric than a central potential, thereby causing a lifting of the accidental degeneracy in the Dirac states for hydrogen. As we have seen above, this calculation is entirely finite—there are no infinities introduced at any stage, and there is therefore no need to introduce any renormalization procedure to explain the data. Thus, in contrast with the use of quantum field theory to explain this effect, the explanation with the elementary interaction field theory is mathematically consistent and the underlying formalism is in terms of field equations with bona fide solutions.

Using the experimental measurement for the Lamb splitting $E(2S_{1/2}) - E(2P_{1/2})$ (which is the most accurately measured value in the states of hydrogenic atoms) given in preceding discussion, we find from the theoretical expression for this splitting [equation (2.2.14)] that

$$g_M = (2{\cdot}087 \pm 0{\cdot}001) \times 10^{-14} \text{ cm} \tag{2.2.16}$$

This is the extra fundamental constant that appears in the elementary interaction field theory. We have seen in the particular application to the hydrogen spectrum that it is in the domains whose radii are this order of magnitude where the conventional Coulomb interaction between charged matter becomes modified. This appears most strikingly in the expression for the wave function [equation (2.2.5)] for the unperturbed hydrogenic electron. It is seen from this form that when the electron–proton separation is of the order of g_M, the modified wave function is significantly different from the Dirac solution. However, as distances greater than this are approached, let us say $r \gtrsim 10^{-13}$ cm, the solution is essentially equal to the standard Dirac solution.

It is perhaps also significant that (a) the magnitude of g_M is close to the Compton wavelength of the proton (i.e. relating to the proton mass) and (b) that the constant $g_{M\gamma}$ [appearing in equation (2.2.2)] is the order of magnitude of the Compton wavelength of the intermediate boson that might be used to describe weak interactions. (The hint in the latter observation is that perhaps weak interactions are not more than a manifestation of electromagnetic interactions at sufficiently small distances. This is also strongly indicated by the fact that the two-component spinor form of the electromagnetic equations is not covariant with respect to reflections in space or time. Such a *formalism* is, of course, a natural one with which to describe the observed weak interactions because of their lack of reflection symmetry.)

Thus far, we have seen that the two measured Lamb splittings in hydrogen, for the states with principal quantum numbers 2 and 3, are predicted by the

elementary interaction theory. The Lamb splitting $[E(4S_{1/2}) - E(4P_{1/2})]$ has not yet been measured. Nevertheless, it is interesting to compare the theoretical values for this splitting according to the present theory and the standard theory (quantum electrodynamics). Using equations (2.2.13c) and (2.2.16) in equation (2.2.11), we find that the present theory gives the result

$$[E(4S_{1/2}) - E(4P_{1/2})]/h = 132.22 \text{ MHz}$$

The result obtained from the conventional quantum electrodynamical formalism is (Petermann, 1958; Layzer, 1961) $133{\cdot}10 \pm 0{\cdot}02$ MHz.

The difference between these two theoretical values ($\sim 0.7\%$) is significantly different than the difference in the previously determined states. Further, as the states with even higher principal quantum numbers are considered, the differences of the predictions of the two formalisms may become increasingly greater. Thus, experimental investigations of the Lamb splittings in the higher states of hydrogen should be quite significant in regard to the test of the predictability of the elementary interaction theory as compared with quantum electrodynamics.

Aside from these differences, however, it is interesting to note that while the two theories, being so different with respect to both formalism and interpretation, should give results for the Lamb splitting that are so close. At the present stage of the theoretical investigation, it can only be conjectured that this is not pure coincidence!

2.3. *Deuterium and He^+*

In view of the general form of the field equations (3.8), Part III, the Lamb splittings for hydrogenic atoms in which the nuclei are D and He should be determined respectively from sets of 3- and 5-coupled nonlinear equations. This, of course, is because of the structures of the respective nuclei. On the other hand, to treat these hydrogenic atoms exactly as we treated normal hydrogen atom, above, would be equivalent to assuming that the D and He nuclei are single particles, with increased mass (for D) and increased mass and charge for He. It is not clear at the present stage that this is an accurate assumption. Nevertheless, should such an assumption be made, the approximation of an infinite nuclear mass (that we made above) would imply that the Lamb splitting in D should be the same as in the *e–p* system. The actual experimental ratio of Lamb splittings in the first excited state $(n + 1) = 2)$ for D : : H is $1{\cdot}0012$. Thus, the error in neglecting the two-body nucleus of D is of the order of $0{\cdot}1\%$.

The error is greater in He^+. In this case, the continued assumption of an infinite nuclear mass, but a nuclear charge that is doubled, leads to a replacement of the fine structure constant γ in equation (2.2.11) (for the Lamb splitting) by the constant $Z\gamma = 2\gamma$. Since the leading term in the derived expression for the Lamb splitting depends on γ^4 (for hydrogen), it follows that the Lamb splitting in He^+ should be $2^4 = 16$ times greater. The experimental ratio for the Lamb splittings (in the first excited state) for He^+

to H is of the order of 14. It is felt by this author that improved predictions for the Lamb splittings in He^+ will follow only after the structure of the helium nucleus is taken into account—by considering better approximations to the solutions of the 5-coupled equations that describe this atomic system.

2.4. *The Lifetimes of the Excited States of Atoms*

Within the elementary interaction approach, we have argued that *spontaneous emission* does not occur. The lifetime of an atom in any state is therefore predicted to be infinitely long—if this atom is the entire system. How, then, does the present theory predict the experimental observations of the lifetimes of the excited states of hydrogen? The answer lies in the theoretical results of our preceding analysis of electron–positron pairs. Recall that it was found that a *bound state* solution for the pair corresponds to the experimental facts that are conventionally interpreted as 'pair annihilation'. But matter is not annihilated here (nor created) at arbitrary times. Since matter *persists*, according to this theory, we were led to the conclusion that any arbitrary region of space must be populated with a large, *but definite number* of such pairs in their ground states of null energy-momentum. This model was used to predict the correct spectral distribution for a radiating black body.

With this model in which a background matter field of particle–anti-particle pairs permeates space, it follows that a box that should contain a hydrogen gas, that is sufficiently rarefied to be able to consider one atom at a time, must still entail the coupling of each of these atoms to the background gas of pairs, as well as its coupling to the walls of the box. Indeed it is just the former coupling that leads to the prediction that an atom in an excited state will decay, at a particular rate. On the other hand, the walls of the box must have a much smaller effect on the lifetimes of excited atoms than would be observable in the laboratory. This follows from the observation that the measured atomic lifetimes seem to be independent of the sizes or shapes of their containers. For example, the lifetimes seem to be the same, whether the atoms are radiating within a star or in a glass tube in the laboratory. Thus, the decay of excited atoms, in the gaseous state, must follow here from the coupling of the individual atomic components of the hydrogen gas to the particle–antiparticle matter fields that we have concluded must be present in the background.

The general expression for the coupling of the excited atom to the background matter field follows from the coupled field equations (3.8) Part III. Two of these equations, in this application to hydrogen, must be taken to have the form given in equation (2.1.1), except for the inclusion here of contributions to the interaction functional $\mathscr{I}^{(k)}$ which relate to the electromagnetic coupling of the hydrogenic electron and proton to the background gas of electron–positron pairs. Similarly, the matter field equations for the pairs [equations (1.10.2)] must now also include the bound electron–proton system in their interaction functionals—i.e. the solutions of equation (2.1.1) must be included in these functionals.

Finally, to take the excited atoms, one at a time, and the pairs in the background matter field, as *separated entities* ,we assume that the coupling is sufficiently weak between all of the constituent two-body systems so that the interaction functionals $\mathscr{I}^{(k)}$ may be represented in terms of an *averaged background potential.*

We have seen in the preceding section that each of the background pairs, when it is in its ground state of null energy-momentum, does not interact electromagnetically, *as a unit.* Nevertheless, we also found that the separate matter field components of the pair do couple electromagnetically to other mutually interacting charged matter. According to the result that was obtained, the action of each pair on the atomic electron is that of two oppositely polarized currents, directly coupled to the atom. It was further shown that in the rest frame of the 'target'—in this case, the excited atom—the effective potential that represents the action of the pair has a form [equation (1.7.4)] of a plane polarized potential field, with associated frequencies, from 0 to 2λ, that run over a continuum of values—because of the continuum of relative velocities between a given pair and the atom.

Taking account of the fact that the effect of $\mathscr{I}_2$ is negligible compared with that of $\mathscr{I}_1$ in this problem, we will ignore the coupling term (2.2.2), but take the proton coupling term (2.1.7), in addition to the effect $\gamma_\nu A_\nu$ of the pairs, where A_ν is the vector potential of the pair, at the site of the atomic electron. We have seen earlier that this has the form:

$$\hat{\mathbf{e}}_k|A|\exp[\pm i(\omega t - \mathbf{k}\cdot\mathbf{r})] \qquad 0 \leqslant \omega \leqslant 2\lambda$$

The polarization orientation (denoted by the unit vector $\hat{\mathbf{e}}_k$) is in a perpendicular direction, relative to the (randomly directed) motion $\mathbf{k}$ of the interaction with the atom. The magnitude of $\mathbf{k}$, of course, is ω/c. The electron coordinate, $\mathbf{r}$, denotes its position relative to its parent nucleus. The average value of $\mathbf{r}$ is then much smaller than the wavelength $2\pi/k = 2\pi c/\omega$, since this is a nonrelativistic gas. With $\mathbf{k}\cdot\mathbf{r} \ll 1$, the factor $\exp(\pm i\mathbf{k}\cdot\mathbf{r})$ may be approximated by unity. This is the 'dipole approximation'. It corresponds to taking the effective vector potential, at the site of the atomic electron, to have the following form

$$\hat{\mathbf{e}}_\mathbf{k}(A_1 \pm iA_2) = \mathbf{e}_\mathbf{k}|A|\exp(\pm i\omega t) \qquad 0 \leqslant \omega \leqslant 2\lambda \qquad (2.4.1)$$

while the other components of the vector potential are zero.

Using the nonrelativistic approximation for the coupled matter field equations [Part III, equation (3.8)] and treating the effect of the pair potential as a small perturbation on the atomic states, the usual expression for the transition probability is obtained from time-dependent perturbation theory (Schiff, 1949). Such a reduction of the formalism clearly follows from the discussion in Part III which demonstrated how the present nonlinear formalism reduces to the ordinary quantum mechanical formalism in the limit of sufficiently small energy-momentum transfer between the interacting components of the physical closed system.

In this limit, the interaction functional $\mathscr{I}_1 \propto \gamma_k A_k$ (in the Dirac form of the theory) reduces to $-iA_k \partial^k$ (in the Schrödinger limit). The vector potential A_k for the pair matter field is, in turn, given in equation (2.4.1) Thus, the coupling of the pair matter fields to the hydrogenic electron in the gas yields the identical formal expression for the probability of transition from one atomic state $|n\rangle$ to another $|n'\rangle$, when time-dependent perturbation theory of ordinary quantum mechanics is utilized. With the electric dipole approximation, this expression has the usual form (Bethe & Salpeter, 1957).†

$$W_{nn'} = \frac{E_{nn'}^3}{2\pi} |\hat{\mathbf{e}}_{\mathbf{k}} \cdot \mathbf{d}_{nn'}|^2 \tag{2.4.2}$$

where

$$E_{nn'} = E_n - E_{n'} = \omega \tag{2.4.3}$$

denotes the energy transfer when a transition occurs. The electric dipole matrix element is

$$d_{nn'} = e \sum_j \langle n' | -i\nabla_j | n \rangle \tag{2.4.4}$$

and $|n\rangle$, $|n'\rangle$ are the limiting Schrödinger states for the electrons of the jth constituent atomic component of the hydrogen gas. The sum in equation (2.4.4) refers to the effect of *one electron–positron pair* on the entire hydrogen gas that is radiating within the container.

Since the only index that distinguishes the different pairs in the background matter field from each other are the *randomly oriented* polarization vectors $\hat{\mathbf{e}}_{\mathbf{k}}$, the total effect of all pairs in the container on the radiating gas can now be taken into account by integrating equation (2.4.2) with respect to all angular variables. After doing this, it is found that the total transition probability that connects the excited state $|n\rangle$ to all other states $|n'\rangle$ has the following form:

$$T_n^{-1} = \tfrac{4}{3}\omega^3 \sum_{n'} |\mathbf{d}_{nn'}|^2 \text{ sec}^{-1} \tag{2.4.5}$$

T_n is the 'lifetime' of the excited state $|n\rangle$. The expression (2.4.5) is, of course, identical with the formula for the lifetime of an excited state as derived from quantum electrodynamics, when the electric dipole approximation is used there.

As we have seen, then, the theoretical expression (2.4.5) for the lifetime of an excited atomic state is not necessarily a consequence of the assumptions of quantum electrodynamics. The same expression for the lifetime is also derivable from ordinary time-dependent perturbation theory, applied to the interaction operator $-iA_k \partial^k$ in the nonrelativistic Schrödinger formalism. The successful predictions of the lifetimes of excited atomic states is then not an absolute test of quantum electrodynamics, nor of ordinary quantum mechanics, nor the present deterministic field theory. The reason, of course, is that the energy-momentum transfer in this type of experiment is sufficiently

† For a complete discussion of such applications to atomic transitions, see Bethe & Salpeter *ibid.*, Sec. 59–63.

low that all three formalisms can be approximated by the same theoretical formulas for the atomic lifetimes.

The salient point in regard to the predictions of the elementary interaction field theory is that the hydrogen gas 'radiates' only by virtue of the transfer of energy and momentum from the excited hydrogen atoms of the gas to the background gas of particle–antiparticle pairs. The reason that the predicted lifetimes are the same in this theory and in the quantum approach is that the vector potential which represents the electromagnetic field of *the pair* (when it is in the particular state of motion discussed in the preceding section) *which acts on the hydrogen atoms*, is *formally* identical with the potential field that would describe a background radiation field of free 'photons' (which is postulated by quantum field theory).

In view of the constancy of the measured lifetimes of atomic states, it is concluded that the density of pairs that form the background 'absorber' is necessarily sufficiently high throughout the portions of the universe that have been probed thus far, that no appreciable effect on the measured lifetimes, due to fluctuations in this density, could have been observed. Nevertheless, the present analysis suggests that if measurements of spectral line shapes (which, in turn, depend on the lifetimes of the excited states) could be carried out with sufficiently high resolution, under more varied conditions of rarefaction and condensation, differences in the lifetimes should occur.

Summing up this section, the correct quantitative predictions for the energy spectrum of hydrogen, *including the Lamb splitting*, as well as the lifetimes of the excited states of hydrogen, follow from an approximation to the deterministic, nonlinear field formalism of the elementary interaction field theory. In contrast with the approach of quantum field theory, (1) the electron–proton system *alone* leads to the correct prediction of the Lamb splitting and (2) the background matter fields for an ideal gas of electron–positron pairs—whose dynamical and kinematic features were derived from an exact solution of the coupled equations for the pair—plays the role of an 'absorber' to describe the decay of excited matter that is conventionally referred to as 'spontaneous decay'. It is also important to emphasize here that the results of the present theory follow from a *finite* description in which perturbative methods are not intrinsic in the formalism—they are rather used only as a convenient method to approximate the solutions of the underlying coupled field equations for a closed system.

Appendix

Computation of the Lamb Splitting

The Lamb splitting was determined in the text [equation (2.2.11)] to be proportional in first order to the number, $|I_+ + I_-|$, where

$$I_\pm = (\eta\gamma)^{-1} \int_0^\infty \frac{F_\pm G_\pm}{\rho^2} \exp(-2\kappa/\rho)\, d\rho \tag{A.1}$$

$\eta^2 = \lambda^2 - E^2$, $\kappa = 16\pi g_M \gamma\eta$ and $F_\pm$, $G_\pm$ are the respective (unperturbed) 'large' and 'small' components of the radial Dirac solutions for hydrogen; the plus and minus signs refer respectively to the states with $K = +1$ and $K = -1$.

The general expressions for the integrals $I_\pm$ will be derived below in terms of a (finite) power series expansion in the fine structure constant γ. While only the first parts of these expansions will be utilized at present, because of the limits of accuracy in the experimental Lamb splittings to be compared with the theory, the derived expansions may be evaluated to any desired accuracy. These expansions in γ are demonstrably convergent since they represent an analytic function in closed form, integrated over all space. The latter is the conserved energy $\int \theta_{00} d^3x$ that follows from the Lagrangian formalism as a consequence of its invariance with respect to time translations [equation (1.1.1) with the exact hydrogenic solutions of equations (2.1.1) inserted]. It should be noted that not only the integrals themselves, but the integrands are finite everywhere. This is because the factor $\exp(-2\kappa/\rho)$ approaches zero faster than any polynomial in $1/\rho$ approaches infinity at the origin.

Consider the Lamb splitting in the hydrogenic state with $n = 1$. This corresponds, in the conventional spectroscopic notation, to the $2S_{1/2} - 2P_{1/2}$ energy separation. In this case, the radial wave functions have the following form in terms of a power series expansion in ρ:

$$\begin{pmatrix} F_\pm \\ G_\pm \end{pmatrix} = \rho^s \sum_{\nu=0}^{1} \begin{pmatrix} a_\nu(\pm) \\ b_\nu(\pm) \end{pmatrix} \rho^\nu e^{-\rho} \tag{A.2}$$

Since $K^2 = 1$,

$$s = [1 - \gamma^2]^{1/2} = 1 - \tfrac{1}{2}\gamma^2 + 0(\gamma^4) \tag{A.3}$$

With the approximation $\rho^s \simeq \rho$ it follows that

$$I_\pm = (\eta\gamma)^{-1}[a_0 b_0 L_0 + (a_0 b_1 + a_1 b_0) L_1 + a_1 b_1 L_2] \tag{A.4}$$

where

$$L_m = (\kappa)^{(m+1)/2} J_m \tag{A.5}$$

and

$$J_m = \int_0^\infty u^m \exp[-2\sqrt{(\kappa)}(u + 1/u)]\, du \tag{A.6}$$

These integrals will be evaluated in the last section of this appendix.

The wave functions (A.2) are determined by the normalization condition and the recursion relations (Schiff, 1949, p. 324) for the coefficients a_ν, b_ν. The normalization condition

$$\eta^{-1} \int_0^\infty (F^2 + G^2) \exp(-2\kappa/\rho)\, d\rho = 1$$

gives, for $n = 1$,

$$\eta = [(a_0^2 + b_0^2) L_2 + 2(a_0 a_1 + b_0 b_1) L_3 + (a_1^2 + b_1^2) L_4] \tag{A.7}$$

and the recursion relations give

$$\begin{aligned} a_1/b_1 &= -\eta/\eta_1, \qquad a_0/b_0 = (s+K)/\gamma \\ b_0/b_1 &= -\frac{2s+1}{2} \frac{\gamma\eta + (s-K)\eta_1}{\gamma\eta + (s-K+1)\eta_1} \end{aligned} \tag{A.8}$$

To order γ^2, equation (A.8) reduces to

$$a_1/b_1 = -\gamma/4$$

$$a_0/b_0 = \begin{matrix} 2/\gamma \\ -\gamma/2 \end{matrix} \qquad K = \begin{matrix} +1 \\ -1 \end{matrix} \qquad b_0/b_1 = \begin{matrix} 3\gamma^2/8 \\ -(1 - 3\gamma^2/8) \end{matrix} \tag{A.9}$$

To the same accuracy,

$$\eta = 1/2r_0 \tag{A.10}$$

where $r_0 = \hbar^2/me^2$ is the 'first Bohr radius'.

We will see below that to this same order of approximation the integrals L_m have the following values (up to $m = 8$):

$$\begin{aligned} &L_0 \cong 1/2 \qquad L_1 \cong L_2 \cong 1/4 \qquad L_3 \cong 3/8 \qquad L_4 \cong 3/4 \\ &L_5 \cong 15/8 \qquad L_6 \cong 45/8 \qquad L_7 \cong 315/16 \qquad L_8 \cong 315/4 \end{aligned} \tag{A.11}$$

With equations (A.11), (A.10), (A.9) and (A.7), the following result, to leading order in γ, is obtained:

$$K = -1 \begin{cases} a_0 = \gamma \sqrt{\dfrac{1}{2r_0}} & b_0 = -\sqrt{\dfrac{2}{r_0}}, \\ a_1 = -\gamma \sqrt{\dfrac{1}{8r_0}} & b_1 = \sqrt{\dfrac{2}{r_0}} \end{cases}$$

$$K = 1 \begin{cases} a_0 = \gamma \sqrt{\dfrac{3}{8r_0}} & b_0 = \gamma^2 \sqrt{\dfrac{2}{32r_0}} \\ a_1 = -\dfrac{\gamma}{4} \sqrt{\dfrac{2}{3r_0}} & b_1 = \sqrt{\dfrac{2}{3r_0}} \end{cases} \tag{A.12}$$

Inserting equations (A.12) and (A.11) into (A.4), we finally obtain the following result:

$$I_+ = 1/6 + 0(\gamma^2) \qquad I_- = -1/2 + 0(\gamma^2)$$

Thus, for the quantum number $n = 1$,

$$|I_+ + I_-|_1 = 1/3 + 0(\gamma^2) \tag{A.13}$$

In a similar way, the $n=2$ solution is characterized by the radial wave function

$$\begin{pmatrix} F_{\pm} \\ G_{\pm} \end{pmatrix} \cong \rho \sum_{\nu=0}^{2} \begin{pmatrix} a_\nu \\ b_\nu \end{pmatrix} \rho^\nu e^{-\rho}$$

so that, for this case,

$$I_{\pm} = (\gamma\eta)^{-1}[a_0 b_0 L_0 + (a_0 b_1 + a_1 b_0) L_1 + (a_0 b_2 + a_2 b_0 + a_1 b_1) L_2 + (a_1 b_2 + a_2 b_1) L_3 + a_2 b_2 L_4] \tag{A.14}$$

The normalization condition gives:

$$\eta = \{(a_0{}^2 + b_0{}^2) L_2 + 2(a_0 a_1 + b_0 b_1) L_3 + [2(a_0 a_2 + b_0 b_2) + (a_1{}^2 + b_1{}^2)] L_4 + 2(a_1 a_2 + b_1 b_2) L_5 + (a_2{}^2 + b_2{}^2) L_6\} \tag{A.15}$$

For this state, $\eta \simeq 1/3r_0$. Using this approximation together with the recursion relations, the following coefficients are obtained for the corresponding wave function, to leading order in γ:

$$K=-1 \begin{cases} a_0 = -\gamma\sqrt{\dfrac{1}{3r_0}} & b_0 = 2\sqrt{\dfrac{1}{3r_0}} \\ a_1 = \dfrac{10\gamma}{9}\sqrt{\dfrac{1}{3r_0}} & b_1 = -4\sqrt{\dfrac{1}{3r_0}} \\ a_2 = \dfrac{-2\gamma}{9}\sqrt{\dfrac{1}{3r_0}} & b_2 = \dfrac{4}{3}\sqrt{\dfrac{1}{3r_0}} \end{cases},$$

$$K=1 \begin{cases} a_0 = \dfrac{-2\gamma}{3}\sqrt{\dfrac{2}{3r_0}} & b_0 = \dfrac{\gamma^2}{3}\sqrt{\dfrac{2}{3r_0}} \\ a_1 = \dfrac{2\gamma}{3}\sqrt{\dfrac{2}{3r_0}} & b_1 = -\dfrac{4}{3}\sqrt{\dfrac{2}{3r_0}} \\ a_2 = \dfrac{-\gamma}{9}\sqrt{\dfrac{2}{3r_0}} & b_2 = \dfrac{2}{3}\sqrt{\dfrac{2}{3r_0}} \end{cases} \tag{A.16}$$

Inserting equation (A.16) into equation (A.14), we find that for $n=2$,

$$|I_+ + I_-|_2 = |\tfrac{1}{9} - \tfrac{1}{3}| = \tfrac{2}{9} + 0(\gamma^2) \tag{A.17}$$

The radial wave function for the $n=3$ state is expressed as the following sum

$$\begin{pmatrix} F_{\pm} \\ G_{\pm} \end{pmatrix} \simeq \rho \sum_{\nu=0}^{3} \begin{pmatrix} a_\nu \\ b_\nu \end{pmatrix} \rho^\nu e^{-\rho}$$

and in this case,

$$I_{\pm} = (\gamma\eta)^{-1}[a_0 b_0 L_0 + (a_0 b_1 + a_1 b_0) L_1 + (a_0 b_2 + a_2 b_0 + a_1 b_1) L_2 + (a_0 b_3 + a_3 b_0 + a_1 b_2 + a_2 b_1) L_3 + (a_1 b_3 + a_3 b_1 + a_2 b_2) L_4 + (a_2 b_3 + a_3 b_2) L_5 + a_3 b_3 L_6] \tag{A.18}$$

The normalization condition for this solution gives the relation:

$$\eta = \{(a_0^2 + b_0^2)L_2 + 2(a_0 a_1 + b_0 b_1)L_3 + [2(a_0 a_2 + b_0 b_2) + (a_1^2 + b_1^2)]L_4 + 2(a_1 a_2 + b_1 b_2 + a_0 a_3 + b_0 b_3)L_5 + [2(a_1 a_3 + b_1 b_3) + (a_2^2 + b_2^2)]L_6 + 2(a_2 a_3 + b_2 b_3)L_7 + (a_3^2 + b_3^2)L_8\} \tag{A.19}$$

In this state, $\eta \sim 1/4r_0$ so that with the recursion relations, the following coefficients are obtained for the radial wave function, to leading order in γ:

$$K=1 \begin{cases} a_0/b_3 = \dfrac{15\gamma}{8} & b_0/b_3 = 15\gamma^2/16 \\ a_1/b_3 = \dfrac{-25\gamma}{8} & b_1/b_3 = 5 \\ a_2/b_3 = \dfrac{5\gamma}{4} & b_2/b_3 = -5 \\ a_3/b_3 = \dfrac{-\gamma}{8} & b_3 = 1/15r_0 \end{cases}$$

$$K=-1 \begin{cases} a_0/b_3 = \dfrac{3\gamma}{2} & b_0/b_3 = -3 \\ a_1/b_3 = \dfrac{-21\gamma}{8} & b_1/b_3 = 9 \\ a_2/b_3 = \dfrac{9\gamma}{8} & b_2/b_3 = -6 \\ a_3/b_3 = \dfrac{-\gamma}{8} & b_3 = 1/(3r_0) \end{cases} \tag{A.20}$$

Inserting equation (A.20) into (A.18), we find that

$$|I_+ + I_-|_3 \cong |\tfrac{1}{12} - \tfrac{1}{4}| = \tfrac{1}{6} + 0(\gamma^2) \tag{A.21}$$

Evaluation of the Integrals J_m

The terms L_m that appear in the terms of the preceding equations depend on the integrals J_m [equation (A.6)] according to the proportionality shown in equation (A.5). Let us now proceed to compute the terms J_m.

It is readily verified that

$$J_{-1} \equiv \int_0^\infty \frac{\exp[-2\sqrt{(\kappa)}(u + 1/u)]}{u} du = 2K_0(4\sqrt{\kappa}) \tag{A.22}$$

where K_0 is the zeroth order modified Bessel function of the second kind (Watson, 1945). Taking derivatives of J_{-1} with respect to $2\sqrt{\kappa}$ will then

generate the other integrals J_m in terms of these derivatives. If we denote the nth derivative of K_0 with respect to its argument by $K_0^{(n)}$, the following relationships are obtained:

$$
\begin{aligned}
J_0 &= -2K_0^{(1)} \\
J_1 &= 4K_0^{(2)} - 2K_0 \\
J_2 &= -8K_0^{(3)} + 6K_0^{(1)} \\
J_3 &= 16K_0^{(4)} - 16K_0^{(2)} + 2K_0 \\
J_4 &= -32K_0^{(5)} + 40K_0^{(3)} - 10K_0^{(1)} \\
J_5 &= 2(32K_0^{(6)} - 48K_0^{(4)} + 18K_0^{(2)} - K_0) \\
J_6 &= 2(-64K_0^{(7)} + 112K_0^{(5)} - 56K_0^{(3)} + 7K_0^{(1)}) \\
J_7 &= 2(128K_0^{(8)} - 256K_0^{(6)} + 160K_0^{(4)} - 32K_0^{(2)} + K_0) \\
J_8 &= -16(32K_0^{(9)} - 72K_0^{(7)} + 54K_0^{(5)} - 15K_0^{(3)} + K_0^{(1)})
\end{aligned}
\tag{A.23}
$$

The exact forms for the derivatives in the preceding equation are as follows:

$$
\begin{aligned}
K_0^{(1)}(z) &= -K_1(z) \\
K_0^{(2)}(z) &= K_0(z) + K_1(z)/z \\
K_0^{(3)}(z) &= -\{(1 + 2/z^2)\, K_1(z) + K_0(z)/z\} \\
K_0^{(4)}(z) &= (3/z^2 + 1)\,(K_0(z) + 2K_1(z)/z) \\
K_0^{(5)}(z) &= -\{2(1 + 6/z^2)\, K_0(z)/z + (24/z^4 + 7/z^2 + 1)\, K_1(z)\} \\
K_0^{(6)}(z) &= (1 + 9/z^2 + 60/z^4)\, K_0(z) + 3(1 + 11/z^2 + 40/z^4)\, K_1(z)/z \\
K_0^{(7)}(z) &= -\{3(1 + 120/z^2 + 17/z^4)\, K_0(z)/z + (1 + 15/z^2 + 192/z^4 \\
&\qquad + 720/z^6)\, K_1(z)\} \\
K_0^{(8)}(z) &= (1 + 18/z^2 + 1272/z^4 + 975/z^6)\, K_0(z) + (4 + 405/z^2 + 1011/z^4 \\
&\qquad + 5040/z^6)\, K_1(z)/z
\end{aligned}
\tag{A.24}
$$

In the problem to which these functions are applied in the text, the argument of the Bessel functions $z = 4\kappa < 10^{-5}$. Thus, to approximate the integrals J_m to one part in 10^5 we will only keep terms to the highest power in $1/z$. According to the power series expansions for the zeroth and first-order Bessel functions, K_0 and K_1, the leading terms are (Watson, 1945)

$$
\begin{aligned}
K_0(z) &= -\ln(z/2 + E) + 0(1/z^2) \qquad (E = \text{Euler's constant}) \\
K_1(z) &= 1/z + 0(1/z^2)
\end{aligned}
\tag{A.25}
$$

With equation (A.21) (keeping terms to the highest power in $1/z$) and equation (A.25) in equation (A.24), we obtain the approximations for the terms L_m that are given in equation (A.11).

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